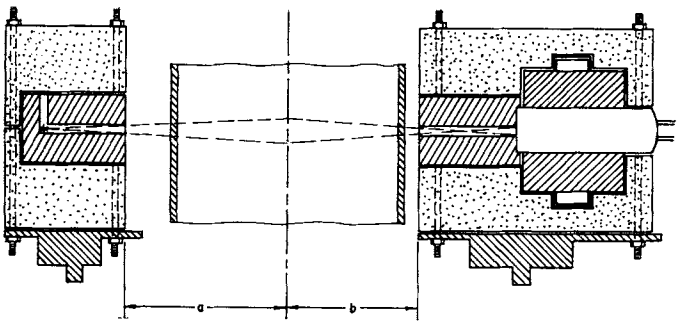

Fig. 10. Scintillation probe and gamma source assembly with the lead shieldings.

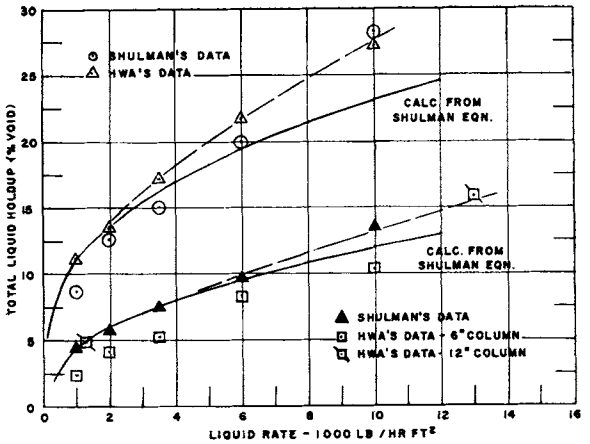

Fig. 11. Comparison of results at $G = 0$ (top two curves for ½-in. packing, lower two curves for 1-in. packing, solid lines for calculated results).

Γ = half life
ϵ = void fraction
λ = wave length
θ = time
Σ = summation

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Manuscript received October 27, 1958; revision received September 2, 1959; paper accepted September 14, 1959. Paper presented at A.I.Ch.E. Atlantic City meeting.

Ion Exchange Kinetics for Systems of Linear Equilibrium Relationships

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The mathematical equations describing the kinetic behavior of ion exchange processes have been solved analytically for systems having linear equilibrium relationships of the type $q^* = K_1 + K_2C^*$. The concentration ratios of the effluent to the influent solution for these cases are found to depend on parameters involving time, position, and relative resistances of the liquid and resin phases. Numerical results have been obtained and are presented in tabular and graphical forms. Furthermore expressions for the constant pattern breakthrough curve for two special cases have also been worked out.

The mathematical equations describing the kinetic behavior of an ion exchange process with the rate-controlling mechanism represented by a combination of resistances for both the liquid and resin phases have been presented elsewhere (*5, 6*). These expressions consist of the equation of continuity, the rate of transfer, the equations describing the concentrations of the resin phase, and the relationship of surface concentrations:

$$\frac{\partial C}{\partial \xi} + \frac{\partial q}{\partial \tau} = 0 \qquad (1)$$

$$\frac{\partial q}{\partial \tau} = k_l (C - C_s) \qquad (2)$$

$$q(\xi, \tau) = \frac{3}{b^3}\int_0^b q_l(r, \xi, \tau) r^2 dr \qquad (3a)$$

$$q_s(\xi, \tau) = q_l(b, \xi, \tau) \qquad (3b)$$

$$q_l(r, \xi, \tau) = \int_0^\tau q_s(\xi, \tau) \frac{\partial}{\partial \tau} H(r, \tau - \lambda) d\lambda \qquad (3c)$$

and

$$q_s = f(C_s) \qquad (4)$$

where

$$\xi = \frac{\rho}{u} z, \quad \tau = t - \frac{\phi}{u} z$$

$$H(r, \tau) = 1 - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\left(\frac{n\pi}{b}\right) r} e^{-D\left(\frac{n\pi}{b}\right)^2 \tau} \cdot \sin \frac{n\pi}{b} r$$

and the following initial and boundary conditions exist:

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$$\left.\begin{matrix} C = 0 \\ q = 0 \\ C_e = 0 \end{matrix}\right\} \text{for } \xi \geq 0, \tau < 0$$

$$C = C_o \text{ at } \xi = 0, \tau \geq 0$$

Equilibrium is assumed to exist at the interface, and consequently Equation (4) represents the equilibrium relationship of the system. It is obvious that the solution corresponding to this set of equations is dependent on the specific form of the equilibrium relationship.

Ion exchange equilibrium has been studied for many years, numerous theories having been proposed to correlate the experimental results. The various attempts to solve this problem fall essentially into two categories, one assuming a process of adsorption and the other applying the law of mass action. Most of the rigorous approaches require a knowledge of the true mechanism involved in the ion exchange process.Unfortunately, for most cases this information is not available. Furthermore the equilibrium relationship formulated on such a basis usually complicates the mathematical equations and gives additional difficulties to the solution of the problem. For this reason a simple empirical equation based directly on experimental data over a restricted concentration range probably would best serve this purpose.

A literature survey on ion exchange equilibrium indicates that if the equilibrium concentration in the resin phase is plotted against that of the liquid phase, several types of relationships result, as presented in Figure 1.

A simple linear relationship passing through the origin can be used to represent curve I. A nonlinear relationship of the Freundlich type is represented by curve II. Curve III can be approximated by a straight line having an intercept on the abscissa. For curve IV the equilibrium concentration in the resin phase increases very rapidly to a certain value and then becomes essentially constant. This relationship is too difficult to fit into a simple equation. However if the concentration range of the initial part of the curve is small, this can be neglected, thus making the concentration in the resin phase constant and independent of the liquid composition. The following expressions can approximate the various cases discussed:

$$q^* = K_1 + K_2 C^* \qquad (5)$$

and

$$q^* = A(C^*)^a \qquad (6)$$

Equation (5) represents the most general linear form and can be used with curves I, III, and IV; Equation (6) approximates curve II. This latter case is not discussed further inasmuch as it has been presented in detail elsewhere (5).

Numerical solutions of Equations (1), (2), (3*a*, *b*, *c*) with Equation (4) represented by Equation (5) with $K_1 = 0$, have already been presented in the literature (4). It is the object of this investigation to seek the solution when the equilibrium relationship is represented by Equation (5).

SOLUTION FOR GENERAL LINEAR CASE

When one substitutes Equation (5) into Equation (3*c*) and carries out the subsequent integration, the expression $q_i(r, \xi, \tau)$ thus obtained is substituted into Equation (3*a*) to give

$$q(\xi,\tau) = K_1 - \frac{6K_1}{b^2}\sum_{n=1}^{\infty}\frac{e^{-D\left(\frac{n\pi}{b}\right)^2\tau}}{\left(\frac{n\pi}{b}\right)^2} + \frac{6DK_2}{b^2}\sum_{n=1}^{\infty}\int_0^{\tau} C_e(\xi,\lambda)e^{-D\left(\frac{n\pi}{b}\right)^2(\tau-\lambda)}\,d\lambda \qquad (7)$$

Differentiating Equation (7) with respect to τ and combining the resultant expression with Equations (1) and (2) follows:

$$\frac{\partial C}{\partial \xi} = -\frac{\partial q}{\partial \tau} = -\frac{6DK_1}{b^2}\sum_{n=1}^{\infty} e^{-D\left(\frac{n\pi}{b}\right)^2\tau} - \frac{6DK_2}{b^2}\sum_{n=1}^{\infty}\int_0^{\tau}\left[\frac{\partial C}{\partial \xi} + \frac{1}{k_1}\frac{\partial^2 C}{\partial \xi \partial \lambda}\right]e^{-D\left(\frac{n\pi}{b}\right)^2(\tau-\lambda)}\cdot d\lambda \qquad (8)$$

The solution of Equation (8) is obtained through the use of Laplace transforms (1). The transform of Equation (8) is found to be

$$\frac{\partial}{\partial \xi}c(\xi,s) = -\frac{6DK_1}{b_2}\sum_{n=1}^{\infty}\frac{1}{s+D\left(\frac{n\pi}{b}\right)^2} - \frac{6DK_2}{b^2}\left[\sum_{n=1}^{\infty}\frac{sc(\xi,s)}{s+D\left(\frac{n\pi}{b}\right)^2} + \sum_{n=1}^{\infty}\frac{1}{k_1}\frac{s}{s+D\left(\frac{n\pi}{b}\right)^2}\frac{\partial}{\partial \xi}c(\xi,s)\right] = \frac{6DK_2}{b^2}\left[-\frac{K_1}{sK_2} - c(\xi,s) - \frac{1}{k_1}\frac{\partial}{\partial \xi}c(\xi,s)\right]\phi_1(s) \qquad (9)$$

where

$$\phi_1(s) = \sum_{n=1}^{\infty}\frac{s}{s+D\left(\frac{n\pi}{b}\right)^2} \qquad (10)$$

By rearrangement one obtains

$$\frac{\partial}{\partial \xi}c(\xi,s)\left[\frac{b^2}{6DK_2\phi_1(s)} + \frac{1}{k_1}\right] + c(\xi,s) = -\frac{K_1}{sK_2} \qquad (11)$$

The solution of this first-order linear differential equation can easily be found to be

$$c(\xi,s) = Be^{-\phi_2(s)\xi} - \frac{K_1}{sK_2} \qquad (12)$$

where

$$\phi_2(s) = \frac{\left[\frac{6DK_2}{b^2}\right]\phi_1(s)}{1 + \frac{1}{k_1}\frac{6DK_2}{b^2}\phi_1(s)} \qquad (13)$$

For the boundary condition of Equation (12)

$$\frac{c(\xi,s)}{C_o} = \frac{1}{s} \text{ at } \xi = 0$$

$$B = \frac{C_o}{s}\left[1 + \frac{K_1}{sK_2}\right]$$

Therefore Equation (12) becomes

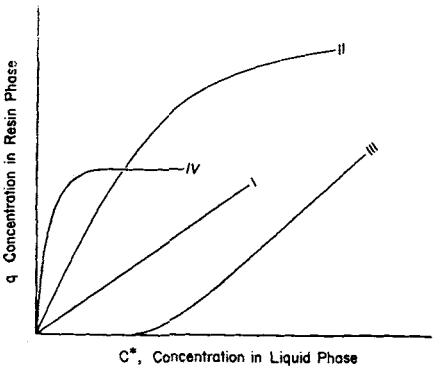

Fig. 1. Graphical representation of typical equilibrium relationships.

TABLE 1. NUMERICAL VALUES OF C/C_o RESULTING FROM EQUATION (15)*

$$\frac{C}{C_o} = -\frac{K_1}{K_2C_o} + \left[1 + \frac{K_1}{K_2C_o}\right]\Phi(\gamma\xi, \sigma\tau, N)$$

for $N = 0$ and $\dfrac{K_1}{K_1L_o} = 0.10$

$\gamma\xi=1$		$\gamma\xi=2$		$\gamma\xi=5$		$\gamma\xi=10$		$\gamma\xi=20$		$\gamma\xi=40$	
$\sigma\tau$	C/C_o	$\sigma\tau$	C/C_o	$\sigma\tau$	C/C_o	$\sigma\tau$	C/C_o	$\sigma\tau$	C/C_o	$\sigma\tau$	C/C_o
0.5	—	0.4	—	1.75	—	4.0	—	1.00	—	20.00	—
0.35	0.172	0.8	0.105	2.50	0.111	5.0	0.056	11.6	0.006	23.20	0.001
0.45	0.306	1.0	0.261	3.00	0.336	6.0	0.261	12.6	0.302	25.20	0.230
0.57	0.452	1.24	0.453	3.25	0.456	6.6	0.460	13.34	0.472	26.68	0.465
0.70	0.584	1.40	0.568	3.50	0.569	7.0	0.686	14.00	0.621	28.00	0.771
0.90	0.771	1.80	0.783	4.50	0.875	8.0	0.826	15.00	0.797	30.00	0.880
1.50	0.946	3.00	0.985	6.00	0.990	9.0	0.944	17.00	0.963	34.00	0.993

* Numerical values of $\Phi(\gamma\xi, \sigma\tau, N)$ are taken from Rosen's work (4).

$$\frac{c(\xi,s)}{C_o} = \left[1 + \frac{K_1}{K_2C_o}\right] e^{-\phi_2(s)\xi} - \frac{K_1}{sK_2} \tag{14}$$

The inverse transform, obtained through the use of the inversion integral formula, is given as

$$\frac{C(\xi,\tau)}{C_o} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{K_1}{K_2C_o}\frac{1}{s} e^{s\tau}\, ds +$$

$$\frac{1}{2\pi i}\int_{a+i\infty}^{a+i\infty}\left[1 - \frac{K_1}{K_2}\right]\frac{1}{s} e^{-\phi_1(s)\xi + s\tau}\, ds =$$

$$-\frac{K_1}{K_2C_o} + \left[1 + \frac{K_1}{K_2C_o}\right]\Phi(\gamma\xi,\sigma\tau,N) \tag{15}$$

$$\Phi(\gamma\xi,\sigma\tau,N) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{s} e^{-\phi_2(s)\xi + s\tau}\, ds \tag{16}$$

It is obvious that $\Phi(\gamma\xi,\sigma\tau,N)$ becomes the same as C/C_o when K_1 vanishes, a situation corresponding to the equilibrium case I. This case represented by Equation (16) has been solved by Rosen (3, 4). The numerical values of Φ, as given by Rosen, are dependent on three dimensionless groups, namely $\gamma\xi$, $\sigma\tau$, and N. In view of these conditions, the expression C/C_o defined by Equation (15) is a function of the parameters K_1/K_2C_o, $\gamma\xi$, $\sigma\tau$, and N. Numerical values for C/C_o corresponding to a special case when $N=0$ and $K_1/K_2C_o=0$ are presented in Table 1 and Figure 2.

CASE OF CONSTANT SURFACE CONCENTRATION

For the special case where $K_2=0$ Equation (5) reduces to $q^* = K_1$. This case corresponds to a situation where the surface concentration of the resin particles is kept constant. Analytical solutions for this special case can be obtained as follows. Referring to Equation (9) one gets

$$\frac{\partial}{\partial\xi} c(\xi,s) = -\frac{6DK_1}{b^2}\sum_{n=1}^{\infty}\frac{1}{s + D\left(\frac{n\pi}{b}\right)^2} \tag{17}$$

The solution of Equation (17) is

$$c(\xi,s) = -\frac{6DK_1}{b^2}\sum_{n=1}^{\infty}\frac{1}{s + D\left(\frac{n\pi}{b}\right)^2}\xi + \text{constant} \tag{18}$$

When one applies the boundary condition, $c(\xi,s)/C_o = 1/s$ at $\xi = 0$, Equation (18) becomes

$$c(\xi,s) = -\frac{6DK_1}{b^2}\sum_{n=1}^{\infty}\frac{1}{s + D\left(\frac{n\pi}{b}\right)^2}\xi + \frac{1}{s} \tag{19}$$

The inverse transform gives

$$\frac{C(\xi,\tau)}{C_o} = 1 - \frac{6DK_1}{C_ob^2}\xi\sum_{n=1}^{\infty} e^{-D\left(\frac{n\pi}{b}\right)^2\tau} = 1 - \nu\xi\sum_{n=1}^{\infty} e^{-D\left(\frac{n\pi}{b}\right)^2\tau} \tag{20}$$

The numerical values of C/C_o as given by Equation (20) depend on two parameters, $(6DK_1)/(C_ob^2)\xi$ and $D\tau/b^2$. The first parameter can be considered to be the product of $\gamma\xi$ and K_1/K_2C_o, while the second parameter is equal to $\sigma\tau/2$. For this case the absence of N can be accounted for, since the surface concentration of the resin phase is constant. Numerical values of C/C_o as expressed by Equation (20) have been computed (6) and are presented in Figure 3.

It can easily be shown that C/C_o as expressed by Equations (15) and (20) could become negative. This corresponds to a physical situation that is impossible. A rationalization for this case is given as follows.

The general linear relationship used for the solutions indicates that there is a definite concentration on the solid surface, even though the liquid phase is free of solute. This condition requires a transfer of material from the liquid phase into the resin phase as long as the resin particles remain unsaturated. Under the restriction of the material balance a transfer of solute from a solute-free solution would give the solution a negative concentration, a situation which is physically impossible. The breakthrough time as predicted with Equations (15) and (20) will be less than the actual case because the resin becomes hypothetically saturated at an earlier time. This effect will become more pronounced with increasing bed height. For this reason a different approach is employed to derive the expressions necessary to predict the breakthrough curve for cases of high values of $\gamma\xi$.

CONSTANT PATTERN BREAKTHROUGH CURVE

It has been shown experimentally that for certain instances the breakthrough curve approaches a constant pattern as the height of the bed increases. This behavior is commonly experienced for systems possessing an equilibrium relationship of the type represented by curve IV. When this condition is established, the breakthrough curve retains a definite shape on a linear plot, subject only to a constant displacement in total elapsed time which is proportional to the bed height. From the material balance one has

$$C_o u\int_o^{\infty}\left[1 - \frac{C}{C_o}\right] dt - z\phi C_o = (z\rho)\, q_{max} \tag{21}$$

The left-hand side of Equation (21) represents the material transferred from the solution; while the right-hand side gives the ultimate capacity of the column.

By rearranging Equation (21) and utilizing the conditions that $(C/C_o), \approx 0$ for $t < t_b$ one obtains

$$\int_{t_b}^{\infty}\left[1 - \frac{C}{C_o}\right] dt = \frac{z\rho}{uC_o} q_{max} - \left[t_b - \frac{\phi}{u} z\right]$$

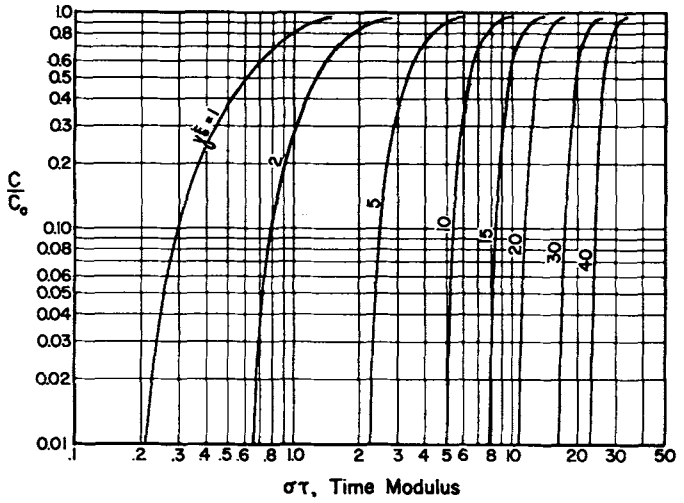

Fig. 2. Concentration ratio C/C_o for the general linear case $N = 0.00$ and $K_1/K_2C_o = 0.10$.

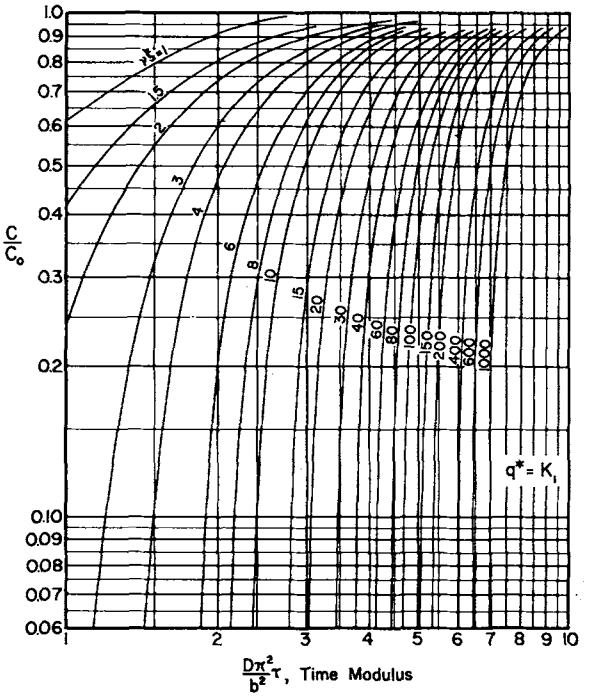

Fig. 3. Concentration ratio C/C_o for the case of constant concentration at the surface ($q^* = K_1$).

or

$$\int_{t_b}^{\infty}\left[1-\frac{C}{C_o}\right]d\tau=\frac{z\rho}{uC_o}q_{max}-\tau_b$$

and

$$\tau_b=\frac{q_{max}}{C_o}\xi-\int_{\tau_b}^{\infty}\left[1-\frac{C}{C_o}\right]d\tau \quad (22)$$

Equation (22) provides a relationship between the breakthrough curve and the breakthrough time from the consideration of a material balance.

It can be shown that for constant values of C/C_o a plot of t vs. z will yield a straight line, and the slope $[\partial t/\partial Z]_{C/C_o}$ is the same for all values of C/C_o, or

$$\left(\frac{\partial t}{\partial z}\right)_{\frac{C}{C_o}}=\left(\frac{\partial t}{\partial Z}\right)_{\frac{C}{C_o}=0}=\left(\frac{\partial t_b}{\partial Z}\right) \quad (23)$$

By transformation Equation (23) becomes

$$\left(\frac{\partial \tau}{\partial \xi}\right)=\left(\frac{\partial \tau_b}{\partial \xi}\right) \quad (24)$$

The value of $dt_b/\partial\xi$ is evaluated by differentiating Equation (22) to give

$$\frac{\partial \tau_b}{\partial \xi}=\frac{q_{max}}{C_o}=\frac{\partial \tau}{\partial \xi} \quad (25)$$

When the constant-pattern condition is established,

$$\frac{\partial(C/C_o)}{\partial \xi}d\xi+\frac{\partial(C/C_o)}{\partial \tau}d\tau=0 \quad (26)$$

which becomes

$$\frac{\partial(C/C_o)}{\partial \xi}=-\frac{\partial(C/C_o)}{\partial \tau}\frac{d\tau}{d\xi}=-\frac{q_{max}}{C_o}\frac{\partial(C/C_o)}{\partial \tau} \quad (27)$$

Also from Equation (1)

$$\frac{\partial(q/q_{max})}{\partial \tau}=-\frac{C_o}{q_{max}}\frac{\partial(C/C_o)}{\partial \xi}=\frac{\partial(C/C_o)}{\partial \tau} \quad (28)$$

Equation (27) gives the obvious conclusion that

$$\frac{q}{q_{max}}=\frac{C}{C_o} \quad (29)$$

for the constant-pattern period.

CONSTANT PATTERN, CONSTANT SURFACE CONCENTRATION, ALL RESISTANCES IN SOLID PHASE

When, the transformation $\chi=\tau-\tau_b$ for the constant-pattern period is introduced, Equation (7) becomes

$$q(\xi,\chi)=K_1-\frac{6K_1}{\pi^2}\sum_{n=1}^{\infty}\frac{e^{-D\left(\frac{n\pi}{b}\right)^2\chi}}{n^2}+\frac{6DK_2}{b^2}\sum_{n=1}^{\infty}\int_o^{\chi}C_s(\xi,\lambda)e^{-D\left(\frac{n\pi}{b}\right)^2(\chi-\lambda)}d\lambda \quad (30)$$

$$\frac{q}{K_1}=\frac{q}{q_{max}}=\frac{C}{C_o}=1-\frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{e^{-D\left(\frac{n\pi}{b}\right)^2\chi}}{n^2} \quad (31)$$

Substituting Equation (31) into Equation (22), one obtains the expression for τ_b:

$$\tau_b=\frac{q_{max}}{C_o}\xi-\frac{b^2}{15D} \quad (32)$$

Equations (31) and (32) completely define the constant-pattern breakthrough curve for the case where the surface concentration is constant. This solution agrees with that obtained by Vermeulen (7) for a similar case.

CONSTANT PATTERN, GENERAL CASE, ALL RESISTANCES IN SOLID PHASE

For the general case where $q_s = K_1 + K_2C_s$, the following expression is obtained by a combination of Equation (7) and Equation (2) and the utilization of the conditions postulated by Equation (29), as well as the definition of χ:

$$\frac{q}{q_{max}}=\frac{C}{C_o}=f(\chi)=\frac{K_1}{q_{max}}\left[1-\frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{e^{-D\left(\frac{n\pi}{b}\right)^2\chi}}{n^2}\right]+\frac{6DK_2L_o}{b^2q_{max}}\int_o^{\chi}f(\lambda)\sum_{n=1}^{\infty}e^{-D\left(\frac{n\pi}{b}\right)^2(\chi-\lambda)}d\lambda-\frac{6DK_2}{b^2k_l}\int_o^{\chi}\frac{\partial f(\lambda)}{\partial\lambda}\sum_{n=1}^{\infty}e^{-D\left(\frac{n\pi}{b}\right)^2(\chi-\lambda)}d\lambda \quad (33)$$

The solution of Equation (33) is obtained by successive approximations (2) as used for the solution of the second integral equation. This is accomplished by the following procedure. When one assumes an arbitrary function $f_o(\chi)$

$$f_1(\chi) = \frac{K_1}{q_{max}}\left[1 - \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{e^{-D\left(\frac{n\pi}{b}\right)^2\chi}}{n^2}\right] + \frac{6DK_2C_o}{b^2q_{max}}\int_o^{\chi} f_0(\lambda)\sum_{n=1}^{\infty} e^{-D\left(\frac{n\pi}{b}\right)^2(\chi-\lambda)}\,d\lambda$$

$$-\frac{6DK_2}{b^2k_l}\int_o^{\chi}\frac{\partial f_0(\lambda)}{\partial\lambda}\sum_{n=1}^{\infty} e^{-D\left(\frac{n\pi}{b}\right)^2(\chi-\lambda)}\,d\lambda$$

$$f_2(\chi) = \frac{K_1}{q_{max}}\left[1 - \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{e^{-D\left(\frac{n\pi}{b}\right)^2\chi}}{n^2}\right] + \frac{6DK_2L_o}{b^2q_{max}}\int_o^{\chi} f_1(\lambda)\sum_{n=1}^{\infty} e^{-D\left(\frac{n\pi}{b}\right)^2(\chi-\lambda)}\,d\lambda$$

$$-\frac{6DK_2}{b^2k_l}\int_o^{\chi}\frac{\partial f_1(\lambda)}{\partial\lambda}\sum_{n=1}^{\infty} e^{-D\left(\frac{n\pi}{b}\right)^2(\chi-\lambda)}\,d\lambda$$

$$f_n(\chi) = \frac{K_1}{q_{max}}\left[1 - \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{e^{-D\left(\frac{n\pi}{b}\right)^2\chi}}{n^2}\right] + \frac{6DK_2C_o}{b^2q_{max}}\int_o^{\chi} f_{n-1}(\lambda)\sum_{n=1}^{\infty} e^{-D\left(\frac{n\pi}{b}\right)^2(\chi-\lambda)}\,d\lambda$$

$$-\frac{6DK_2}{b^2k_l}\int_o^{\chi}\frac{\partial f_{n-1}(\lambda)}{\partial\lambda}\sum_{n=1}^{\infty} e^{-D\left(\frac{n\pi}{b}\right)^2(\chi-\lambda)}\,d\lambda$$

if it is assumed that

$$\lim_{n\to\infty} f_n(\chi) = f(\chi) \qquad (34)$$

It is apparent that this procedure becomes tedious; however for the special case where diffusion in the resin phase is dominant, $N \to 0$, the second integral of Equation (33) vanishes, and under these conditions the approximate expression for $f(\chi)$ is found to be

$$\frac{C}{C_o} \approx 1 - \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{e^{-D\frac{\pi^2}{b^2}\left[n^2 - \frac{6K_2L_o}{\pi^2q_{max}}\right]\chi}}{n^2} \qquad (35)$$

Furthermore the expression for τ_b is

$$\tau_b = \frac{q_{max}}{C_o}\xi - \frac{6b^2}{D\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^2\left[n^2 - \frac{6K_2C_o}{\pi^4q_{max}}\right]} \qquad (36)$$

Similar to the previous case, Equations (36) and (38) predict the breakthrough curve for the general case when diffusion in the resin phase controls.

The infinite series of Equation (36) can be obtained by the following expression:

$$\sum_{n=1}^{\infty}\frac{1}{n^2\left[n^2 - \frac{6K_2L_o}{\pi^2q_{max}}\right]} = \sum_{n=1}^{\infty}\frac{1}{n^4} + a^2\sum_{n=1}^{\infty}\frac{1}{n^6} + a^4\sum_{n=1}^{\infty}\frac{1}{n^8} + a^6\sum_{n=1}^{\infty}\frac{1}{n^{10}} + \ldots \qquad (37)$$

where $a^2 = (6K_2C_o)/(\pi^2q_{max})$. The sum of the series $\Sigma\,(1/n^{2r})$, where r is a positive integer, is presented in Table 2.

The developments of this investigation make possible the establishment of the breakthrough curve for systems having linear equilibrium relationships. For small values of $\gamma\xi$ Equations (14) and (20) apply. For larger values the equations for the constant-pattern breakthrough curve would give better results should such a curve exist.

NOTATION

b = radius of particle, cm.

c = Laplace transform of C

C = concentration of solution, meq./cc.

C_o = concentration of influent solution, meq./cc.

C_s = concentration of solution at surface, meq./cc.

D = diffusion coefficient in resin phase, sq. cm./sec.

k_l = mass transfer coefficient for liquid film, meq./(sec.)(g. of dry resin)(meq./cc.)

K_1, K_2 = constants for Equation (5)

N = $(3DK_2)/(b^2k_l)$

q = average concentration in resin, meq./g. of dry resin

q_l = point concentration in resin, meq./g. of dry resin

q_{max} = concentration in resin phase in equilibrium with influent solution, meq./g. of dry resin

q_s = concentration on resin surface, meq./g. of dry resin

t = time, sec.

t_b = breakthrough time, sec.

u = superficial velocity, cm./sec.

z = bed height, cm.

Greek Letters

γ = $(3DK_2)/(b^2)$

ν = $(6DK)/(C_ob^2)$

ξ = $(\rho z)/u$

ρ = bulk density of resin, g. of dry resin/cc.

σ = $(2D)/(b^2)$

τ = $t - (\phi z)/u$

τ_b = $t_b - (\phi z)/u$

ϕ = porosity

χ = $\tau - \tau_b$

TABLE 2. NUMERICAL VALUES OF THE INFINITE SERIES, $\sum_{n=1}^{\infty}\frac{1}{n^{2r}}$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \qquad \sum_{n=1}^{\infty}\frac{1}{n^8} = \frac{2^6}{9!}\cdot\frac{3}{5}\pi^8$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90} \qquad \sum_{n=1}^{\infty}\frac{1}{n^{10}} = \frac{2^8}{11!}\cdot\frac{5}{3}\pi^{10}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{2^4}{7!}\cdot\frac{\pi^6}{3} \qquad \sum_{n=1}^{\infty}\frac{1}{n^{12}} = \frac{2^{10}}{13!}\cdot\frac{691}{105}\pi^{12}$$

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Manuscript received January 26, 1959; revision received July 7, 1959; paper accepted July 10, 1959.